NASA TT F-58

IN-02

TECHNICAL TRANSLATION

F-58

UNSTEADY MOTION OF A WING OF FINITE SPAN IN A COMPRESSIBLE MEDIUM

By E. A. Krasilshchikova

Translated from Izvestiya Akademii Nauk, Otdelenie Tekhnicheskikh Nauk SSSR (Moscow), no. 3, Mar. 1958.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON May 1961

lH

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL TRANSLATION F-58

UNSTEADY MOTION OF A WING OF FINITE SPAN IN A COMPRESSIBLE MEDIUM*

By E. A. Krasilshchikova

F
5
8

Under consideration is the turbulent motion of a compressible fluid caused by the unsteady movement of a thin wing of finite span, moving according to a given law (refs. 1 and 2).

For the solution of the boundary problems the method which we developed earlier in connection with our investigation of plane parallel unsteady movements of a fluid (ref. 3) is used.

The article gives the solution of the problem in quadratures for all forms of unsteady movement of a wing, in the case when the basic velocity of the wing's movement is supersonic, and when the end effect or the influence of the whirl system spreading behind the wing is not affecting the wing's surface.

1. We shall consider the movement of a thin, slightly bent wing of finite span with a small angle of attack.

We shall assume that the basic movement of the wing is a gradual rectilinear movement with a generally variable velocity taking place within an unlimited volume of compressible fluid coming to rest at infinity. Let us impose on the basic movement of the wing additional small unsteady movements in the course of which the surface of the wing can be deformed.

We shall use a right-handed rectilinear coordinate system of Oxyz-axes, invariably linked to the space in which the wing's movement is taking place. We aim the Ox-axis in the direction of the wing's movement and we place the xOy-plane in such a way that the z-coordinates of the points of the wing's surface are small. (See figs. 1 and 2.)

The law of the wing's basic movement will be considered to be given in the form

$$x = F(t) \tag{1.1}$$

*Translated from Izvestiya Akademii Nauk, Otdelenie Tekhnicheskikh Nauk SSSR (Moscow), no. 3, Mar. 1958, pp. 25-32.

where F is an arbitrary continuous function of time, and where the x-coordinate, for purposes of definiteness, will be the coordinate of some fixed point C on the leading edge of the wing.

The normal velocity component on both sides of the wing's surface will be subject to the law

$$\sigma_n = A \tag{1.2}$$

where A is a point-time function on the wing's surface defined by F

$$A = A_0 + A_1 \qquad \left(A_0 = -F'(t)\beta\right) \tag{1.3}$$

5
8

The functions β (the angle of attack of the elements of the wing) and A are given at every point of the wing's surface. These are small arbitrary integrable functions of their arguments. The first summand in the expression for A represents the basic movement of the wing; the second represents additional unsteady movements.

We shall assume the flow of the fluid to be irrotational and to take place in the absence of external forces. The velocity potential φ of the perturbed flow of the fluid and its derivatives will be considered as small magnitudes of the first order, and small magnitudes of the second order will be disregarded. Under these assumptions, as it is known, the velocity potential satisfies the wave equation, which, in fixed coordinate axes, has the form

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \frac{1}{a^2}\varphi_{tt} = 0 \tag{1.4}$$

where a is the velocity of sound in an unperturbed medium.

We shall establish the boundary conditions satisfied by the function φ and by its derivatives. We will shift the boundary conditions on the wing's surface parallel to the Oz-axis onto the projection $\sum_0$ of the wing on the fixed xOy-plane, which is equivalent to ignoring small magnitudes of the second order. Thus on the basis of the given law (eq. 1.2)) for the normal velocity components of the points of the wing's surface, we get the streamlining, or downwash, condition*

---

*NASA reviewer's note: The downwash, and hence the source strength, is assumed to be known on the wing surface and in unperturbed areas; when needed elsewhere, however, it must be determined by solutions to integral equations. No new solutions are derived from the formal treatment.

$$\varphi_z = A(x,y,t) \tag{1.5}$$

which must be fulfilled on both the upper and lower sides of $\sum$.

From the surface of the wing, in the direction opposite to its movement, there descends a vortex surface, known as the vortex sheet, on which the velocity potential, as on the wing's surface itself, is subject to a break in continuity. The projection $\sum_1$ of the vortex F 5 8 sheet on the xOy-plane is a semistrip extending from the trailing edge of the wing in the direction opposite to the Ox-axis. On the whirl surface the kinematic condition expressing the continuity of the normal velocity component of the fluid's particle must be satisfied, and also the dynamic condition expressing the continuity of pressure must be satisfied. Since on the vortex surface the direction of the normal deviates little from the direction of the Oz-axis, we will also shift the boundary conditions parallel to the Oz-axis onto the projection of the vortex surface on the xOy-plane, which again amounts to disregarding small magnitudes of the second order.

From the continuity of pressure, it results that in the region $\sum_1$ the derivative function

$$\varphi_t = 0 \tag{1.6}$$

It follows from the same conditions that everywhere in the xOy-plane, but outside the regions $\sum$ and $\sum_1$, where the medium is perturbed, the velocity potential is equal to zero

$$\varphi = 0 \tag{1.7}$$

If the velocity of the wing's basic movement is supersonic, that is, $F'(t) > a$, then the medium is perturbed only in that part of the space which is restricted by Mach wave. Outside this wave, in the xOy-plane, the condition

$$\varphi_z = 0 \tag{1.8}$$

must also be satisfied. In addition to this, the Chaplypin-Zhukovski principle must be observed on the trailing edge of the wing at every moment. Thus, the boundary problem consists of finding a function $\varphi(x,y,z,t)$ that satisfies equation (1.4), the boundary conditions (1.5), (1.6), (1.7), (1.8), and the following conditions pertaining to its derivatives:

$$\lim_{r\to\infty} \varphi_x = \lim_{r\to\infty} \varphi_y = \lim_{r\to\infty} \varphi_z = 0 \quad \text{where} \quad r = \sqrt{x^2 + y^2 + z^2} \tag{1.9}$$

It is sufficient to solve the problem for the upper half-space of the wing's movement. The velocity potential for the lower half-space will be found from the condition

$$\varphi(x,y,-z,t) = -\varphi(x,y,z,t) \tag{1.10}$$

since the function φ is an odd function relative to the z-coordinate, F
when the movement of the wing is rectilinear. 5 8

2. We shall consider the solutions (ref. 5, chs. 1 and 3)

$$\varphi(x,y,z,t) = \frac{f(\xi,\eta,\tau)}{\sqrt{(x - \xi)^2 - k^2(y - \eta)^2 - k^2z^2}} \tag{2.1}$$

$$k = \sqrt{\frac{u_1^2}{a^2} - 1}. \quad \tau = t - \frac{u_1(x - \xi)}{u_1^2 - a^2} + \frac{a}{u_1^2 - a^2}\sqrt{(x - \xi)^2 - k^2(y - \eta)^2 - k^2z^2} \tag{2.2}$$

of the equation

$$\left(u_1^2 - a^2\right)\varphi_{1xx} - a^2\varphi_{1yy} - a^2\varphi_{1zz} + \varphi_{1tt} + 2u_1\varphi_{1xt} = 0 \tag{2.3}$$

where f is an arbitrary function of its arguments. The magnitude u_1 is an arbitrary parameter of equation (2.3). Formula (2.1) shows that the variables ξ, η, and τ satisfy the equation

$$\left(u_1^2 - a^2\right)(t - \tau)^2 + (x - \xi)^2 + (y - \eta)^2 + z^2 - 2u_1(x - \xi)(t - \tau) = 0 \tag{2.4}$$

Equation (2.4) is the equation of the surface, which can be obtained as the intersection of the characteristic surface

$$\left(u_1^2 - a^2\right)(t - \tau)^2 + (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 - 2u_1(x - \xi)(t - \tau) = 0 \tag{2.5}$$

of equation (2.3) with the hyperplane $\zeta = 0$.

We shall place at all points $M(\xi,\eta)$ of the plane xOy sources with potentials of the form (2.1). Because of the linearity of equation (2.3), its solution is a function expressed by the formula

$$\varphi_1(x,y,z,t) = \iint\limits_{S_\infty} \frac{f(\xi,\eta,\tau)}{\sqrt{(x-\xi)^2 - k^2(y-\eta)^2 - k^2z^2}}\, d\eta\, d\xi \tag{2.6}$$

F 5 8

By making the following change of variable:

$$\eta = y - \frac{1}{k}\sqrt{(x-\xi)^2 - k^2z^2}\cos\theta \tag{2.7}$$

we transform the integral (2.6), taking into consideration solutions (2.2), into the form

$$\varphi_1(x,y,z,t) = \frac{1}{k}\iint\limits_{S_\infty} f\left\{\xi\left(y - \frac{1}{k}\sqrt{(x-\xi)^2 - k^2z^2}\cos\theta\right)\right.$$

$$\left.\left(t - \frac{u_1(x-\xi)}{u_1^2 - a^2} + \frac{a}{u_1^2 - a^2}\sqrt{(x-\xi)^2 - k^2z^2}\sin\theta\right)\right\} d\theta\, d\xi \tag{2.8}$$

We note that in the area of integration S_∞ which represents, generally speaking, the entire xOy-plane, a bi-connected region S' can always be singled out, so that the variables of integration vary between the limits $c' \leqq \xi \leqq x - kz$ and $x + kz \leqq \xi \leqq c''$; $\left(y - k^{-1}\sqrt{(x-\xi)^2 - k^2z^2}\right) \leqq \eta \leqq y + k^{-1} - \sqrt{(x-\xi)^2 - k^2z^2}$ or $0 \leqq \theta \leqq \pi$, where c' and c'' are constant magnitudes, satisfying the inequalities $c' > x + Kz$, respectively. In the remaining part $S_\infty - S'$ of the region S_∞, the limits of integration will either not depend on z at all, or will depend on z only in the combination k^2z^2.

If we differentiate equation (2.8) with respect to the parameter z, and consider the value of this derivative when $z = 0$, we will obtain the relation (ref. 5):

$$f(x,y,t) = -\frac{1}{2\pi}\varphi_{1z}(x,y,0,t) \tag{2.9}$$

Thus for any value of time t, formula (2.6) establishes the dependence between the function φ_1 at an arbitrary point of the xyz-space and the derivative φ_{1z}, normal to the xOy-plane.

In particular, in solution (2.6), if the parameter u_1 is assumed to be equal to zero, and the notation $\left|\varphi_1(x,y,z,t)\right|_{u_1=0} = \varphi(x,y,z,t)$ is introduced, we shall have the solution of equation (1.4) in the form (refs. 5 and 6)

$$\varphi(x,y,z,t) = -\frac{1}{2\pi} \iint_{S_\infty} \frac{\varphi_z\left\{\xi,\eta,0,t - a^{-1}\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}\right\}}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}}\, d\eta\, d\xi \tag{2.10}$$

where the variables of integration vary between the limits $-\infty \leqq \xi \leqq +\infty$ and $-\infty \leqq \eta \leqq +\infty$.

3. We shall consider the space of the variables x, y, and t. We break up the xyt-space into the regions V, V_1, and V_2 (see fig. 3). We shall define each of these regions as in reference 4.

We shall take, on the surface of the wing, an arbitrary point N defined by the coordinates x_{1n} and y_{1n} in a mobile coordinate system $Ox_1y_1z_1$, which is invariably related to the moving wing, where

$$x_1 = x - F(t), \quad y_1 = y, \quad z_1 = z$$

(See figs. 1 and 2.) Let the curve NN' represent the law of movement

$$x = F(t) + x_{1n}, \quad y = y_{1n}$$

of the point N (fig. 3). At every point of this curve the streamlining condition is fulfilled. The aggregate of curves representing the laws of movement of the whole set of points of the wing's surface forms a three-dimensional region V. The region V is bounded by the surface $\sum^*$. The surface $\sum^*$ is the locus of the curves representing the law of movement of the wing's contour points.

Let the curve ACBD, which forms the wing's contour (fig. 2) in a mobile coordinate system, be given by the equation

$$y_1 = \Psi(x_1) \tag{3.1}$$

Then the equation of the surface $\sum^*$ will have the form

$$y = \Psi\left[x - F(t)\right] = \Psi'(x,t) \tag{3.2}$$

In the region V the derivative φ_z is given according to the streamlining condition (1.5).

The region V_1 is partly bounded by the surface $\sum^*$, which is formed by the curves representing the laws of movement of the points on the trailing edge of the wing, the arc ABD of the wing's contour, and by two planes which are tangent to the surface $\sum^*$ along the curves AA' and BB'. The curves AA' and BB', respectively, represent the laws of movement of the points A and B, that is, the extreme left and extreme right points of the wing's contour (fig. 3). The tangent planes are not represented in the diagram.

In the region V_1, the boundary condition (1.6) is satisfied.

The region V_2 is the remaining part of the xyt-space, confined in the interior of the region $V + V_1$. In the region V_2 the condition (1.7) is satisfied.

In figure 2, the plane region $\sum$ represents the projection of the wing on the xOy-plane, at some instant of time t_1 (fig. 3). The region $\sum_1$ is the projection of the vortex sheet on the xOy-plane at the same instant of time t_1. The plane region $\sum_1$ and, consequently, also the region V_1 extend to infinity, provided the movement is of unlimited duration. If the movement starts from rest, then these regions are bounded.

Represented in figure 4 are the plane regions $\sum'$, $\sum'_1$, and $\sum'_2$, obtained from the intersection of the xOt-plane with the V-, V_1-, V_2-planes, respectively. The curve CC' represents the law of movement of the point C of the wing's contour in the xOt-plane, and the curve DD' represents the law of movement of the point D of the wing's contour. The equation of the curve DD' is

$$x = F(t) - l$$

where l is a chord of the center plane.

In the investigation of various boundary problem variants, an important part is played by conics defined by equation (3.3):

$$(x - \xi)^2 + (y - \eta)^2 - a^2(t - \tau)^2 = 0 \tag{3.3}$$

This family of conics (3.3) can be obtained by intersecting the char- F
acteristic surfaces of equation (1.4) with the hyperplane $\xi = 0$ when 5
$z = 0$. The tangent of the half angle at the vertex of these conics is 8
equal to the velocity of sound, that is, the propagation velocity of small perturbations in the compressible medium. The branches of the family of conics (3.3) corresponding to the values $\tau > t$ will be called, for brevity's sake, Ω-conics (fig. 5).

The bending surface of the family of Ω-conics with vertices on the surface $\sum^*$ is a boundary of the region in the xyt-space, corresponding to the perturbed state of the medium in the xOy-plane of the wing's movement.

The family of Ω-conics performs the same function as the family of straight lines X_1 and X_2, given in the construction of solutions for plane problems (ref. 3, p. 28) by the equations

$$\xi + a\tau = C_1$$

$$\xi - a\tau = C_2$$

respectively.

The dotted lines in figure 4 represent the straight lines which are the lines of intersection of the Ω-conics with the xOt-plane.

We shall apply solution (2.10) to the boundary problem, and determine the derivative φ_z from the boundary conditions of the problem, given in the xyt-space. We shall introduce in the integral (2.10) the element dS of the surface of the hyperboloid defined by the equation

$$(x - \xi)^2 + (y - \eta)^2 + z^2 - a^2(t - \tau)^2 = 0 \tag{3.4}$$

which is satisfied by the variables ξ, η and

$$\tau = t - a^{-1}\sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}$$

under the sign of the function φ_z. By introducing the element

$$dS = \sqrt{EG - F_1}\, d\eta\, d\xi$$

(E, G, and F_1 being the coefficients in the first main quadratic form), we shall represent solution (2.10) in the form

F 5 8

$$\varphi(x,y,z,t) = -\frac{a}{2\pi} \iint_{S(x,y,z,t)} \frac{\varphi_z\left(\xi,\eta,0,t - a^{-1}\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}\right)}{\sqrt{\left(1 + a^2\right)(x-\xi)^2 + \left(1 + a^2\right)(y-\eta)^2 + a^2z^2}}\, dS \qquad (3.5)^*$$

where we integrate over the surface S. Surface S is the surface of the branch of the hyperboloid (3.4), extending to infinity in the direction of decreasing values of time. Integration along the branch of the hyperboloid extending in the direction of increasing values of time has no physical meaning.

To every aggregate of variables x, y, z, and t there corresponds a surface S with vertex at the point P(x,y,t - (z/a)), represented in figure 3.

In those cases where the surface S intersects only the region V and that part of the xyt-space which corresponds to the unperturbed state of the medium in the xOy-plane, the derivative φ_z is known everywhere on the surface. In order for formula (3.5) to correspond to the solution of the formulated problem, the derivative φ_z must be taken from the boundary conditions (1.5) and (1.8) without resorting to the construction of integral equations.

In the general case when the set of variables x, y, z, and t is arbitrary, the surface S intersects the regions V, V_1, and V_2, and consequently, the derivative φ_z turns out to be unknown on a part of surface S. In order for formula (3.5) to correspond to the solution of the formulated problem, the unknown derivative φ_z must be found from the integral equations constructed on the basis of the boundary conditions (1.6) and (1.7).

It is convenient in the actual computation of the quadratures and in the construction of the integral equations to pass from the surface integral (3.5) to a double integral with a plane area of integration.

---

*NASA reviewer's note: Equation (3.5) and, consequently, the expression for dS on page 11, as well as equation (3.6), appear to be incorrectly given.

The region S_1 is a part of the xOt-plane, which is bounded on the right by the hyperbola $a^2(t - \tau)^2 - (x - \xi)^2 - z^2 = 0$ (see fig. 4). If condition (3.9) holds, then the left boundary of the region S_1 is the curve which is the projection on the xOt-plane of the line of intersection of the surface $\sum^*$ with the surface of the hyperboloid S, in that part where the current coordinate $\eta < y$. Analogously, the region S_2 is determined as that region in the xOt-plane which corresponds to that part of the surface of the hyperboloid S where $\eta > y$.

F

If in formula (3.11), the derivative φ_z is considered a function 5
of two variables ξ and τ, we shall get a formula (ref. 3, 8
formula (1.10)) for the determination of the velocity potential in the case when the wing is of infinite span.

In particular, the curve forming the wing's contour can consist of segments of smooth curves; for example, it can be piecewise smooth.

In particular, the curve which describes the law of the wing's movement can be given not by one equation $x = F(t)$, but can consist of segments of smooth curves given by various equations. The main velocity of the wing's movement can vary stepwise; that is, the derivative F' can have discontinuities of the first type, which correspond to the angular points on the curves describing the laws of movement of the wing's points. The envelope of the Ω-conics with vertices at the angular points corresponding to the same instant of time divides the xyt-space into regions having various analytic modes of solution of the problem. The envelope of the Ω-conics with vertices at the points on the curves which represent the laws of movement of angular points, or the points of junction of the wing's contour, divides the xyt-space into regions having various analytic solutions of the problem.

In particular, in the process of the wing's movement, the character of the additional unsteady movements of the wing can vary in such a way that the points of the wing's surface can be included as separate degrees of freedom in the additional movements; steady movements may alternate with unsteady ones, etc. In all these cases, from the point of view of the form of the given function $A(x,y,t)$ the region V is divided by the given surfaces into a series of regions.

For instance, let the wing move according to the law $x = F(t)$ and let there be no additional movements; that is, on the wing's surface the derivative

$$\varphi_z = F'(t)\beta(x,y,t) = \overline{A}$$

We shall assume that at the instant of time t_0 all points of the wing's surface begin to effect harmonic fluctuations of the wing. Starting from the instant of time t_0, on the wing's surface

$$\varphi_z = -F'(t)\beta(x,y,t) + R_e A_1(x,y,t) \exp i\omega t = A$$

The plane region A_0, D_0, B_0, C_0 is obtained as the result of the intersection of the region V with the plane $\tau = t_0$ which divides the region V into two parts, namely, when

$$\tau < t_0, \quad \varphi_z = \overline{A}$$

and when

$$\tau > t_0, \quad \varphi_z = \overline{\overline{A}}$$

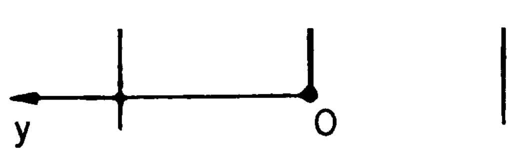
Let the wing move according to the law $x = F(t)$. We will assume that starting from the instant of time t_0, the points of the wing's surface are progressively included in the additional fluctuating movements with the relative velocity $v(t)$ directed along the chord of the wing in the direction of its movement. The surface given by the equation

$$\xi = \int_0^{\tau} \left[F'(\tau) + v(\tau)\right] d\tau + \text{constant} = f_0(\tau)$$

also divides the region V into two parts. One part is to the left of this surface, where $\varphi_z = \overline{A}$, the other is to the right of this surface where $\varphi_z = \overline{\overline{A}}$ (fig. 7).

The curve A_1, D_0, B_1, C_1 is obtained by intersecting the surface $\sum^*$ with the surface given by the equation $\xi = f_0(\tau)$.

The results also retain force when in the process of the wing's movement the area of the carrier surface changes to a finite magnitude, which is reflected in the form of the given boundary $\sum^*$ of the region V (ref. 3, p. 35). This may be the case in the operation of some form of mechanization of the wing. In the process of the actual computation of quadratures in accordance with formula (3.5), the region of integration S must be split into component parts, depending on the form of



the given derivative φ_z in the region V, and depending on the form of the surface $\sum^*$, whose separate parts can be given by various equations.

Translated by Consultants Custom Translations, Inc.,

16

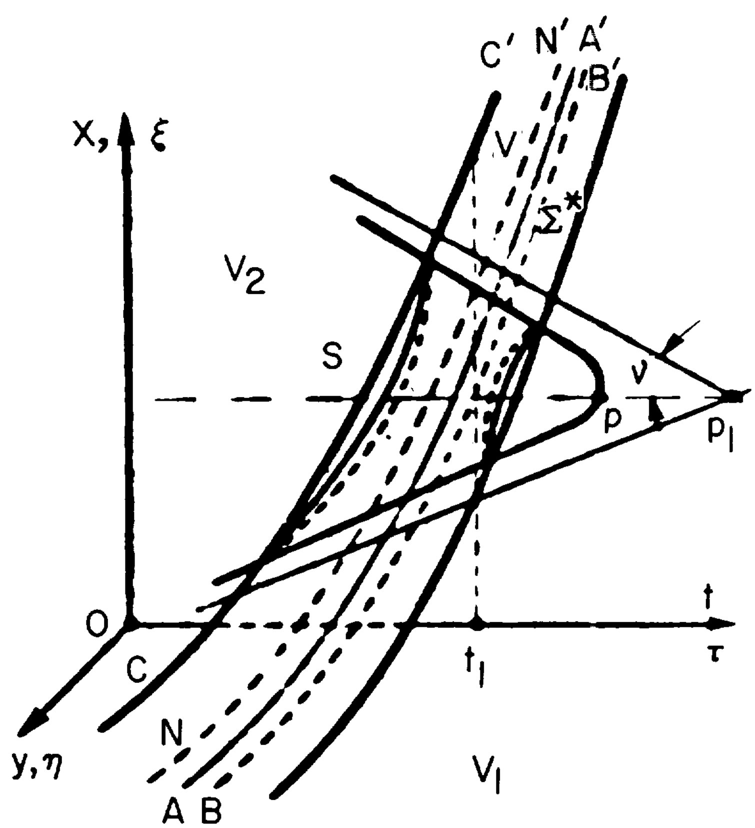


Figure 3.

F-58

F-58

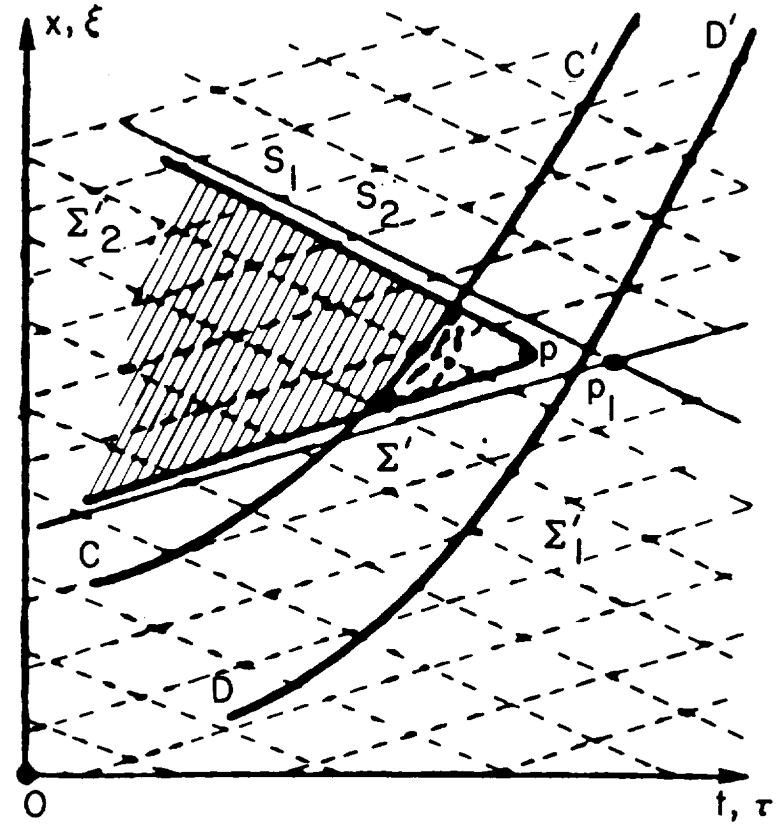


Figure 4.

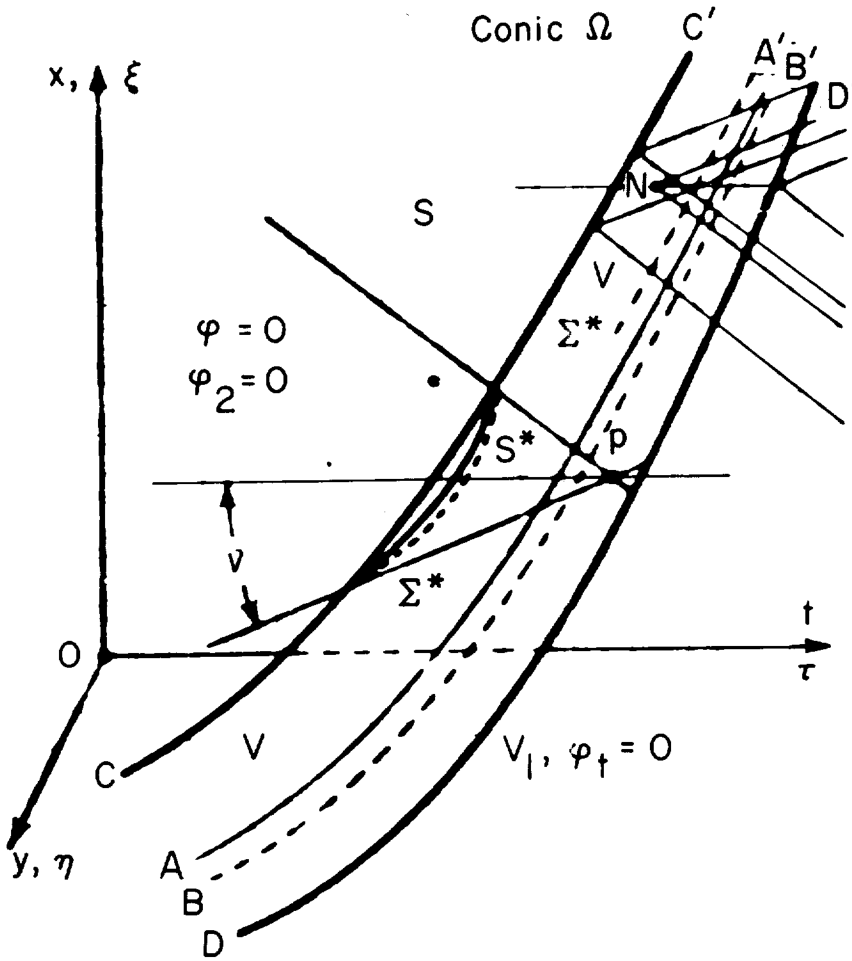


Figure 5.

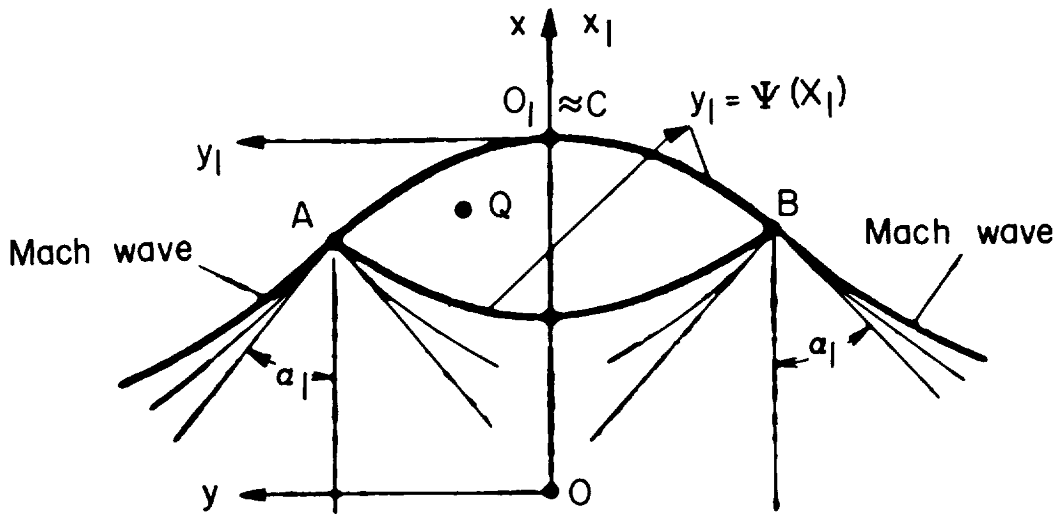


Figure 6.

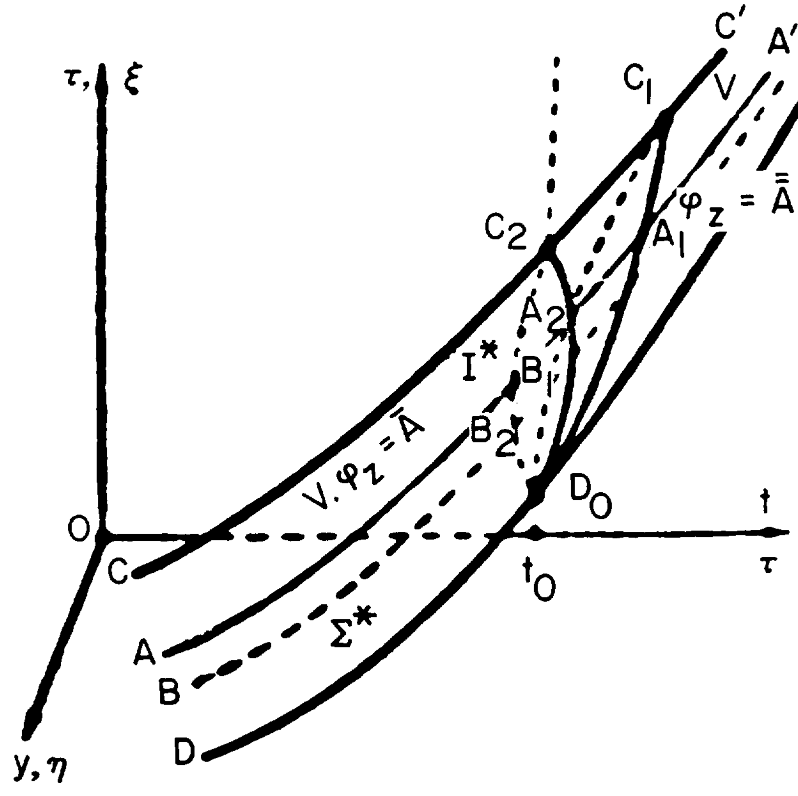


Figure 7.

F-58